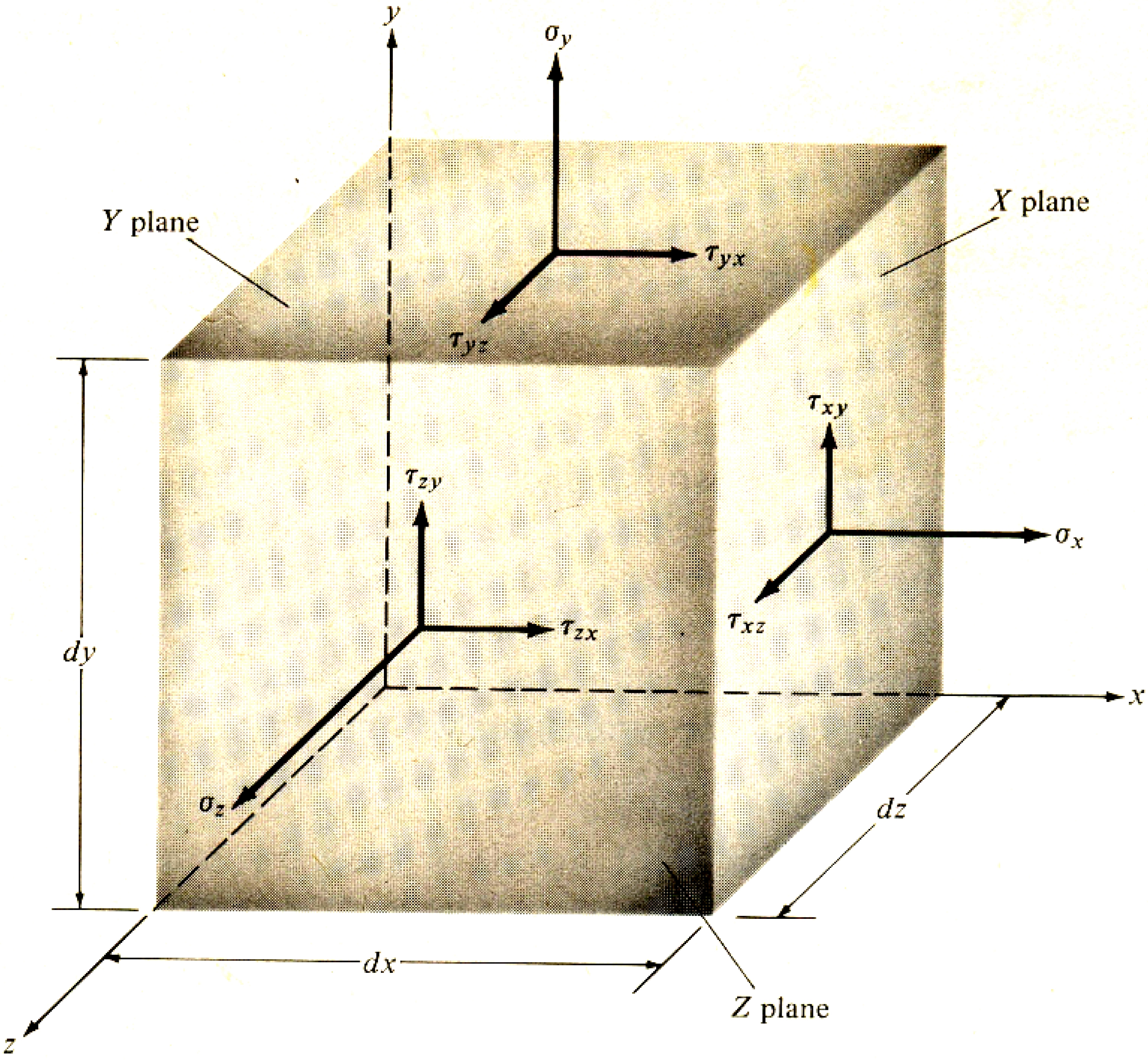
Biaxial Stress System

The more general stress condition in which the stresses on an element acts in both x and y directions and all stress components in z- direction vanish is known as biaxial stress system. It is different from one dimensional or uniaxial stress condition considered in the previous section. Thus it is assumed that all the components in the z-direction ($\tau_{xz} = \tau_{yz} = \sigma_z = 0$) shown in Fig. No.1 are zero. The stress element shown in Fig. No. 2 is obtained; it is the most general condition that can exist. Biaxial stresses arise in the analysis of beams, pressure vessels, shafts and many other structural members.



y
σy
Y plane
X plane
τyx
τyz
τxy
τzy
σx
τxz
τzx
dy
x
σz
dz
dx
Z plane
z

FIGURE NO. 1

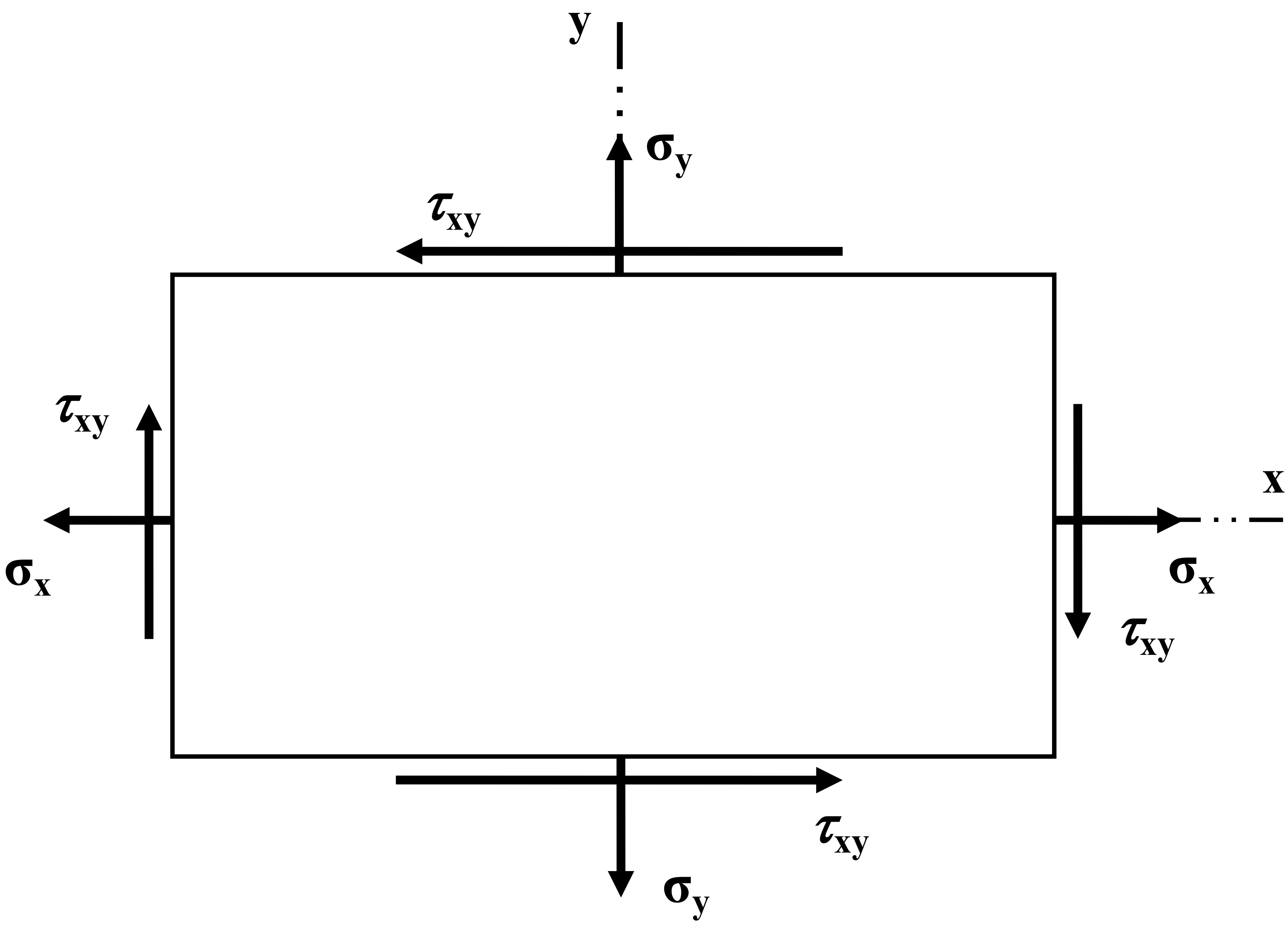


FIGURE NO. 2

A class of common engineering problems involving stresses in a thin plate or free surface of a structural element, such as the surfaces of thin walled pressure vessels under external or internal pressure, the free surface of shafts in torsion or beams under transverse loads.

General case of two dimensional stress.

In general if a plane element is removed from a body it will be subjected to the normal stresses σ_x and σ_y together with the shearing stress τ_{xy} as shown in fig 3. It is desirable to investigate the state of stress on any inclined plane t defined by angle θ, positive ccw as shown in fig 4. The resulting relations would make it possible to determine

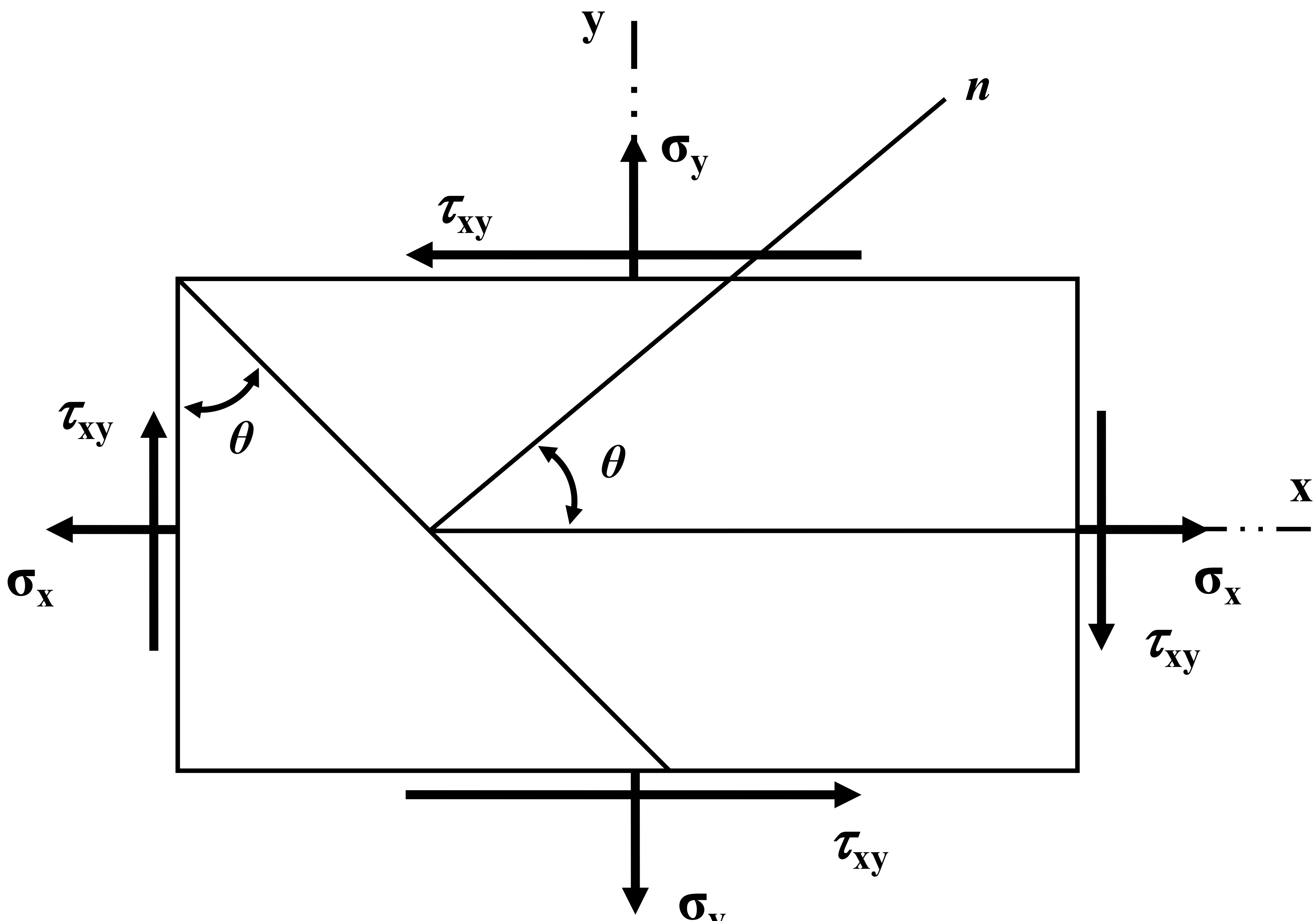


FIGURE NO. 3

normal stress σ_n and the shear stress τ on the inclined plane t form normal n to the inclined plane t form knowledge of normal and shear stresses on the X and Y planes (usually known). Note that the normal n to the inclined plane t make an angle θ with the x-axis. This angle θ is considered to be positive when measured in the ccw direction from the positive end of the x-axis. According to sign convention, which is the one used throughout this text, a shear stress is positive if it produces clockwise rotation of the element on which it is acting and negative if it produces counterclockwise rotation. Thus τ_{xy} on the X plane in Figure is positive while τ_{xy} on the Y plane (i.e., τ_{yx}) is negative. However, as was stated earlier, normal stresses are positive if tensile and negative if compressive. Isolate the small wedge to the left of the inclined plane and construct its free-body diagram as shown in Figure (4). If one assumes the area of the inclined plane to be *A* square units, then the area of the X plane would be A cos θ square units and the area of the Y plane would be A sin θ square units, as shown in Figure (5). Thus the forces on the three faces of the wedge produced by the normal and shear stresses acting on them can be determined in terms of A and the trigonometric functions of the angle θ, as shown in Figure (5).

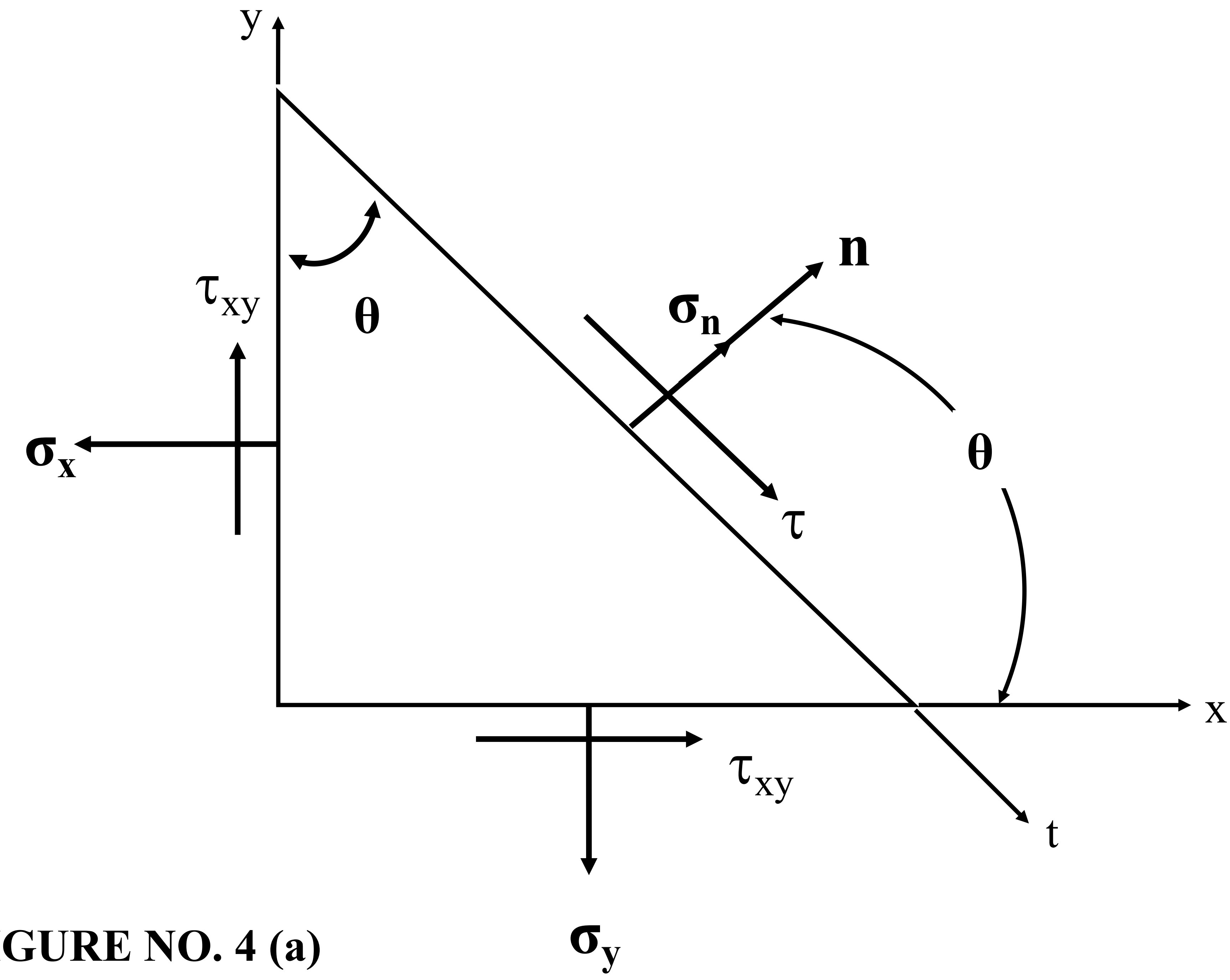


FIGURE NO. 4 (a)

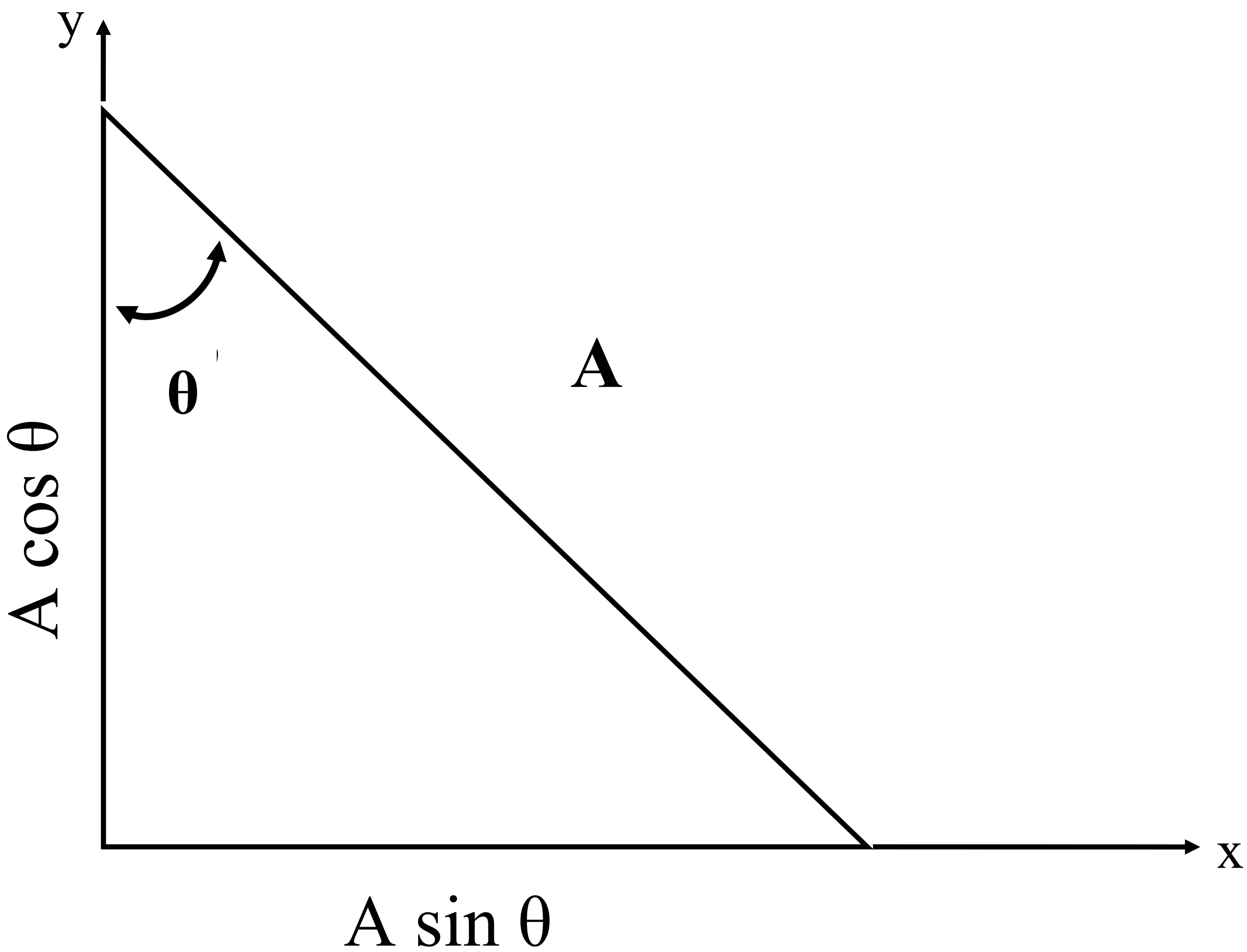


FIGURE NO. 4 (b)

Evidently the desired stresses acing on the inclined planes are internal quantities with respect to the element shown in Fig. 4. Following the usual procedure of introducing a cutting plane so as to render the desired quantities external to the new section, the originally rectangular element is cut along the plane inclined at the angle θ to the x-axis and thus obtain the triangular element shown in Fig. 5. Since half of the material have been removed in the rectangular element, it has been replaced it by the effect that it exerted upon the remaining lower triangle shown and this effect in general consists of both normal and shearing forces acting along the inclined plane.

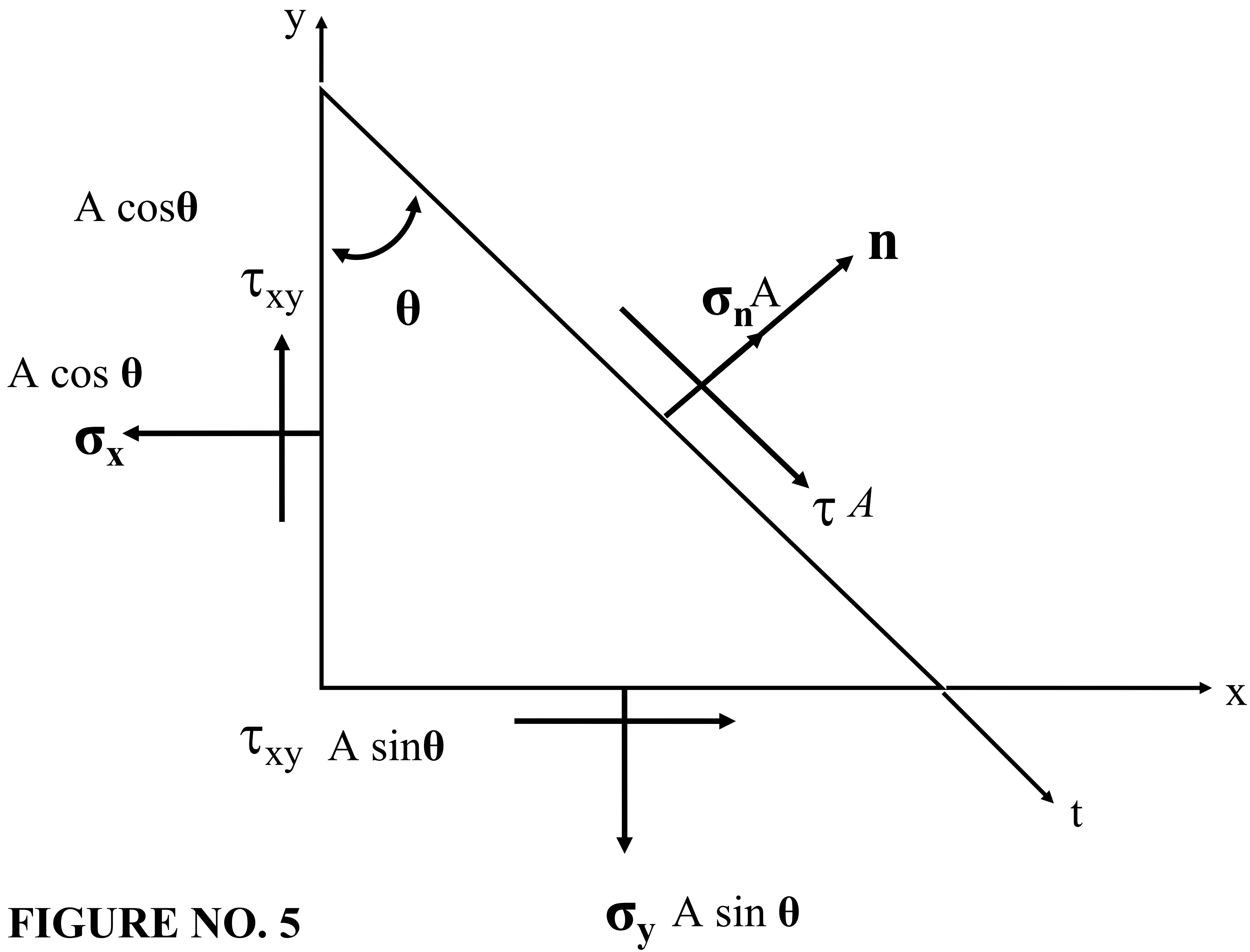

y
A cosθ
τ_{xy}
θ
n
$\sigma_n A$
A cos θ
σ_x
τ A
x
τ_{xy} A sinθ
t
σ_y A sin θ

FIGURE NO. 5

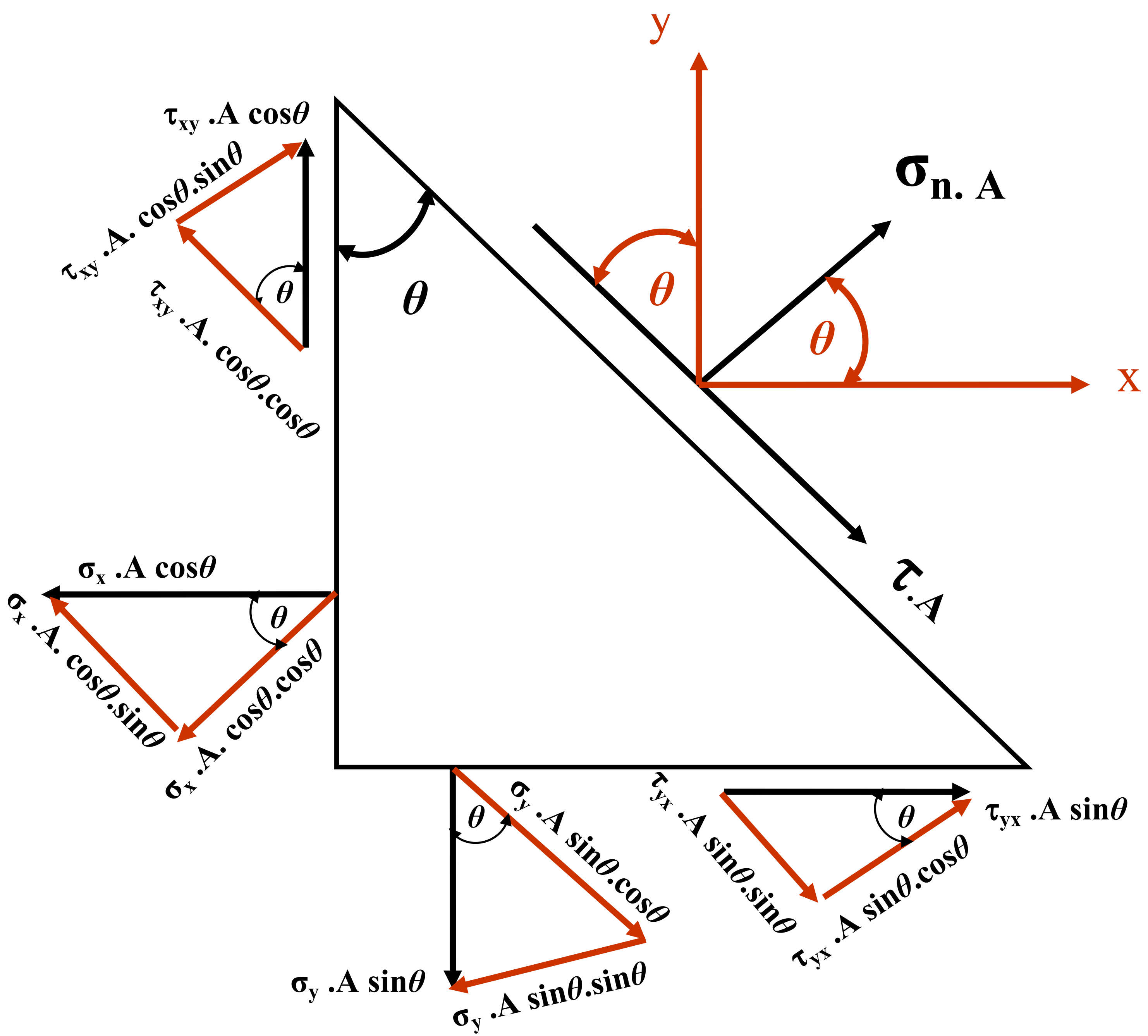

y
x
σn. A
τ.A
θ
τxy .A cosθ
τxy .A. cosθ.sinθ
τxy .A. cosθ.cosθ
σx .A cosθ
σx .A. cosθ.sinθ
σx .A. cosθ.cosθ
σy .A sinθ
σy .A sinθ.cosθ
σy .A sinθ.sinθ
τyx .A sinθ
τyx .A sinθ.sinθ
τyx .A sinθ.cosθ

FIGURE NO. 6

Magnitudes of the normal and shearing stresses corresponding to these forces have been designated by σ_n and τ respectively as discussed earlier. Thus our problem reduces to finding the unknown stresses σ_n and τ in terms of the known stresses σ_x, σ_y, and τ_{xy}. It is to be carefully noted that the free-body diagram, Fig. 4. indicates stresses acting on the various faces of the element, and not forces. Each of these stresses is assumed to be uniformly distributed over the area on which it acts.

Summation of forces in Fig. along the direction of σ_n $\Sigma F_n = 0$.

$$\sigma_n A = (\sigma_x A\cos\theta)\cos\theta + (\sigma_y A\sin\theta)\sin\theta - (\tau_{yx} A\sin\theta)\cos\theta - (\tau_{xy} A\cos\theta)\sin\theta$$

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta - 2\tau_{xy}\sin\theta\cos\theta$$

$$\sigma_n = \sigma_x\left(\frac{1+\cos 2\theta}{2}\right) + \sigma_y\left(\frac{1-\cos 2\theta}{2}\right) - 2\tau_{xy}\frac{\sin 2\theta}{2}$$

$$\sigma_n = \frac{\sigma_x}{2} + \frac{\sigma_x.\cos 2\theta}{2} + \frac{\sigma_y}{2} - \frac{\sigma_y.\cos 2\theta}{2} - \tau_{xy}\sin 2\theta$$

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau_{xy}\sin 2\theta \qquad \rightarrow (1)$$

Summing the forces along τ (t-direction) Σft = 0.

Shearing Stress acting parallel to the inclined Plan.

$$A.\tau = (\sigma_x A\cos\,\theta)\sin\theta - (\sigma_y .A\sin\theta)\cos - (\tau_{yx} A\sin\theta)\sin\theta + (\tau_{xy} .A\cos\,\theta)\cos\,\theta$$

$$\tau = \sigma_x \cos\,\theta\sin\theta - \sigma_y \sin\theta\cos\,\theta - \tau_{yx}\sin^2\theta + \tau_{xy}\cos^2\theta$$

Consider an element

$$\tau = \left(\sigma_x - \sigma_y\right)\sin\theta\cos\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$

$$\tau = \frac{\left(\sigma_x - \sigma_y\right)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad \rightarrow (2)$$

The stresses σ'$_x$ and τ' acting on a plane at right angle to the inclined plane on which σ$_x$ and τ act are obtained by substituting $(\theta + \frac{\pi}{2})$ for θ in 1 & 2.

Then these equation are as follows

$$\sigma'_n = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \quad \rightarrow (3)$$

$$\tau' = -\left(\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + t_{xy}\cos 2\theta\right) \qquad \rightarrow (4)$$

By combining 3 & 4 the resulting equation is

$$\sigma_n + \sigma'_n = \sigma_x + \sigma_y \qquad \rightarrow (5)$$

This shows that sum of normal stresses on any perpendicular planes is constant. Similarly comparison of B & B' results is

$$\tau' = -\tau \qquad \rightarrow (6)$$

This shows that shear stresses on perpendicular planes are equal in magnitude but opposite in sign.

Note>> $\mathrm{Cos}(2\theta+180) = -\mathrm{Cos}(2\theta)$

$\mathrm{Sin}(2\theta+180) = -\mathrm{Sin}(2\theta)$

Principal Stresses and location of angles on which that act.

In the design and stress analysis, the maximum stresses are desired in order to ensure safety of the load carrying member. The above equations can be used if we know at what angle θ it occurred. The angle θ can be found by differentiating the function and setting the result equal to zero.

$$\frac{d\sigma_n}{d\theta} = 0$$

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\frac{d\sigma_n}{d\theta} = 0 + \frac{\sigma_x - \sigma_y}{2}(-\sin 2\theta)(2) - (\tau_{xy} \cos 2\theta)(2)$$

$$0 = -(\sigma_x - \sigma_y)\sin 2\theta - 2\tau_{xy} \cos 2\theta$$

$$0 = -(\sigma_x - \sigma_y)\frac{\sin 2\theta}{\cos 2\theta} - 2\tau_{xy} \frac{\cos 2\theta}{\cos 2\theta}$$

$$0 = -(\sigma_x - \sigma_y)\tan 2\theta - 2\tau_{xy}$$

$$\tan 2\theta_P = \frac{-2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \rightarrow (7)$$

$$\theta_P = \frac{1}{2}\tan^{-1}\left(\frac{-2\tau_{xy}}{\sigma_x - \sigma_y}\right) \qquad \rightarrow (8)$$

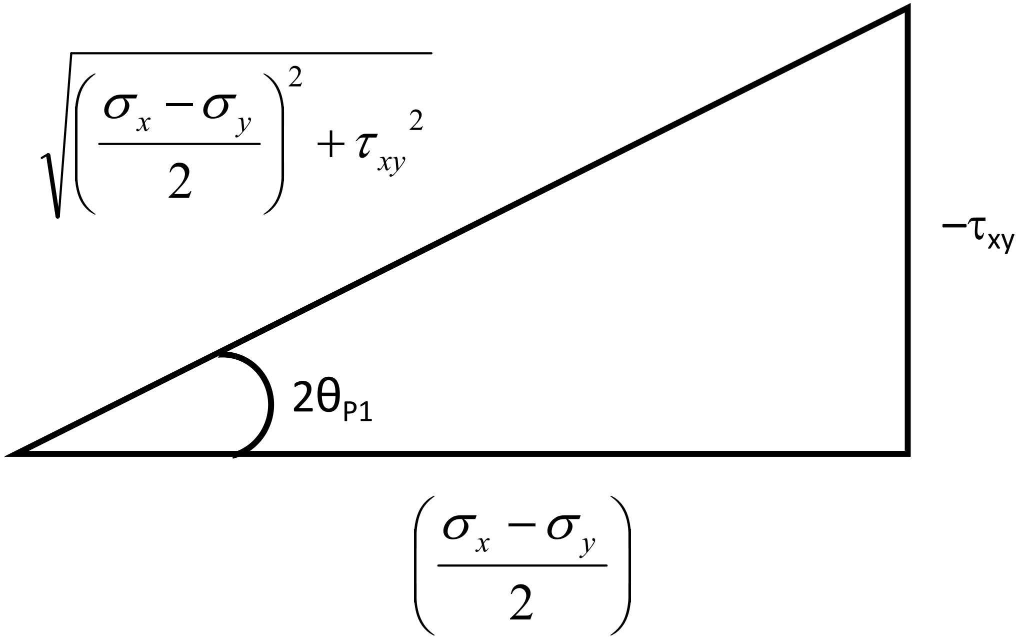
Equations 7 & 8 define the planes of maximum and minimum normal stresses.

Planes defined by equation 7 and 8 are known as principal planes. Thus there are two solutions of equation 7, and two values of $2\theta p$ (differing by 180^o) and also two values of θp (differing by 90^o).

The normal stresses that exist on these planes are called principal stresses. On one of these two planes, the normal stress is maximum and on the second the normal stress is minimum. It will be further shown that the Principal Plane are free from shear stresses and therefore another way of defining is that the Principal stress are the normal stresses acting on the planes on which shear stresses are zero.

Values of maximum and minimum normal stresses can be obtained by substituting values of θP from 7 or 8 into 1. In order to get $\cos 2\theta$ and $\sin 2\theta$ a right angle triangle is drawn as shown in Fig 7 (a & b).

The values of $\sin 2\theta p$ and $\cos 2\theta p$ as found from the above two diagram may now be substituted in equation (1) to get maximum and minimum values of normal stresses.



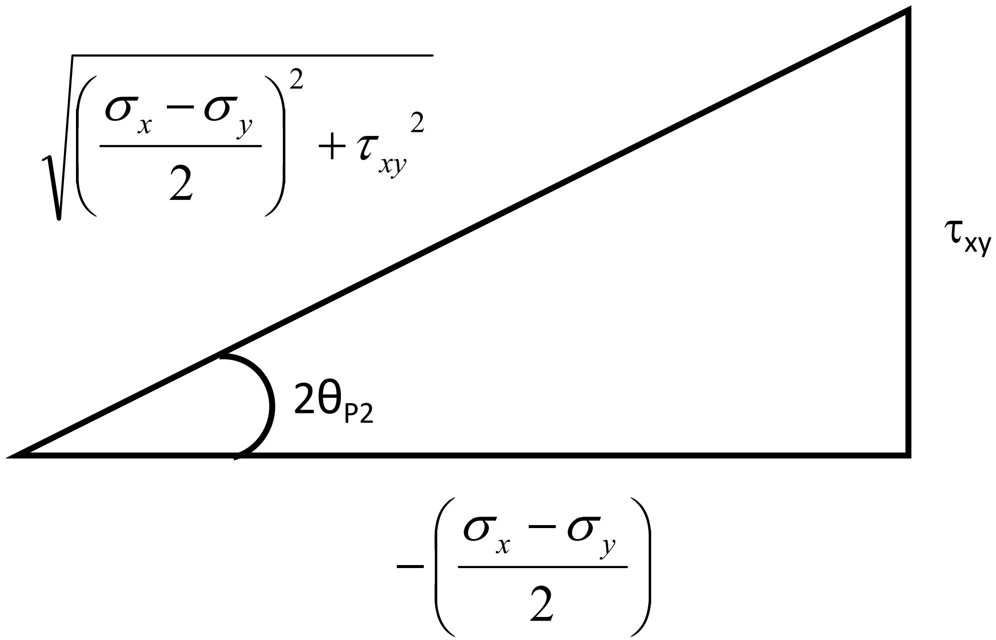


FIGURE NO. 7 (a, b)

$$\cos 2\theta p = \frac{\pm \dfrac{\sigma_x - \sigma_y}{2}}{\sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\;2}}}$$

$$\cos 2\theta p_1 = \frac{\dfrac{\sigma_x - \sigma_y}{2}}{\sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\;2}}}$$

$$\cos 2\theta p_2 = \frac{-\left(\dfrac{\sigma_x - \sigma_y}{2}\right)}{\sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\;2}}}$$

$$\sin 2\theta p = \frac{\mp \tau_{xy}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

$$\sin 2\theta p_1 = \frac{-\tau_{xy}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

$$\sin 2\theta p_2 = \frac{\tau_{xy}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}}$$

Substituting these values in A

$$\sigma_{\max} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\left[\frac{\frac{\sigma_x - \sigma_y}{2}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}}\right] - \frac{\tau_{xy}(-\tau_{xy})}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}}$$

$$\sigma_{\max} = \frac{\sigma_x + \sigma_y}{2} + \frac{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}} + \frac{(\tau_{xy})^2}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}}$$

$$\sigma_{\max} = \frac{\sigma_x + \sigma_x}{2} + \frac{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}}$$

$$\sigma_{\max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} \qquad \rightarrow (9)$$

$$\sigma_{\min} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\left[\frac{-\left(\frac{\sigma_x - \sigma_y}{2}\right)}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}}\right] - \frac{\tau_{xy}.(\tau_{xy})}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}}$$

$$\sigma_{\min} = \frac{\sigma_x + \sigma_y}{2} - \frac{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}} - \frac{\tau_{xy}{}^2}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}}$$

$$\sigma_{\min} = \frac{\sigma_x + \sigma_y}{2} - \left(\frac{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}} \right)$$

$$\sigma_{\min} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} \qquad \rightarrow (10)$$

Normally these Principal stress are denoted by σ_1 & σ_2.

$$\sigma_1 = \quad \sigma_{\max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} \qquad \rightarrow (11)$$

$$\sigma_2 = \sigma_{\min} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} \rightarrow (12)$$

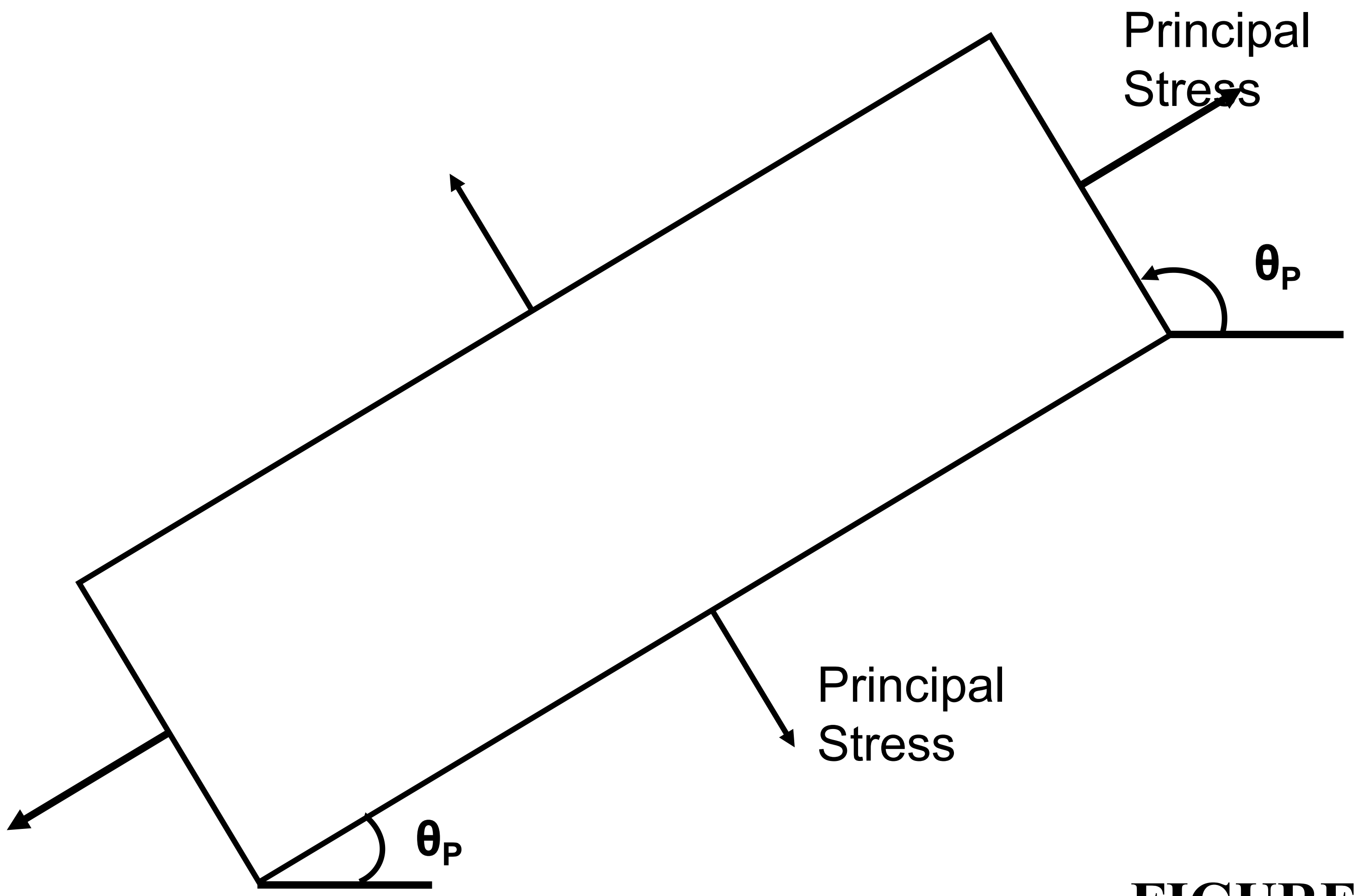


FIGURE NO. 8

The stresses given by 11 and 12 are the principal stresses and they occur the on the principal plane. By substituting one of the values of θp from equation 7 into equation 1 one may readily determine which of the two principal stresses is acting on that plane. The other Principal stress naturally acts on the other Principal plane.

Another useful relation is obtained by adding the values of Principal stresses

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y \qquad \rightarrow (13)$$

It states that sum of the normal stresses on any two orthogonal planes through a point in a stressed body is constant.

Shear stress is zero at principal plane.

The vanishing of the shear stress on Principal planes can be shown if the values of cos2θp and sin2θp is substituted in equation 2.

$$\tau = \frac{\sigma_x - \sigma_y}{2}.\sin 2\theta + \tau_{xy}\cos 2\theta \qquad (2)$$

$$\tau = \left(\frac{\sigma_x - \sigma_y}{2}\right)\left(\frac{-\tau_{xy}}{\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}}\right) + \tau_{xy}.\frac{\sigma_x - \sigma_y}{2\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2}}}$$

$$\tau = 0$$

It can be concluded that on the element on which principal stresses act the shear stress is zero regardless of the values of σ_x, σ_y, τ_{xy}.

Maximum shearing stress and its location.

There are certain values of angles θ that leads to maximum value of shear stress for a given set of stresses σ_x, σ_y, τ_{xy}, the orientation of the planes on which maximum shear stress can be found using the same technique i.e. differentiating equation 2 w.r.t. θ and setting it equal to zero.

$$\tau = \frac{\sigma_x - \sigma_y}{2}.\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\frac{d\tau}{d\theta} = \frac{\sigma_x - \sigma_y}{2}(\cos 2\theta)(2) + \tau_{xy}(-\sin 2\theta)(2) = 0$$

$$\textit{dividing by } \cos 2\theta$$

$$\tau_{xy}(\sin 2\theta).2 = \sigma_x - \sigma_y.\cos 2\theta$$

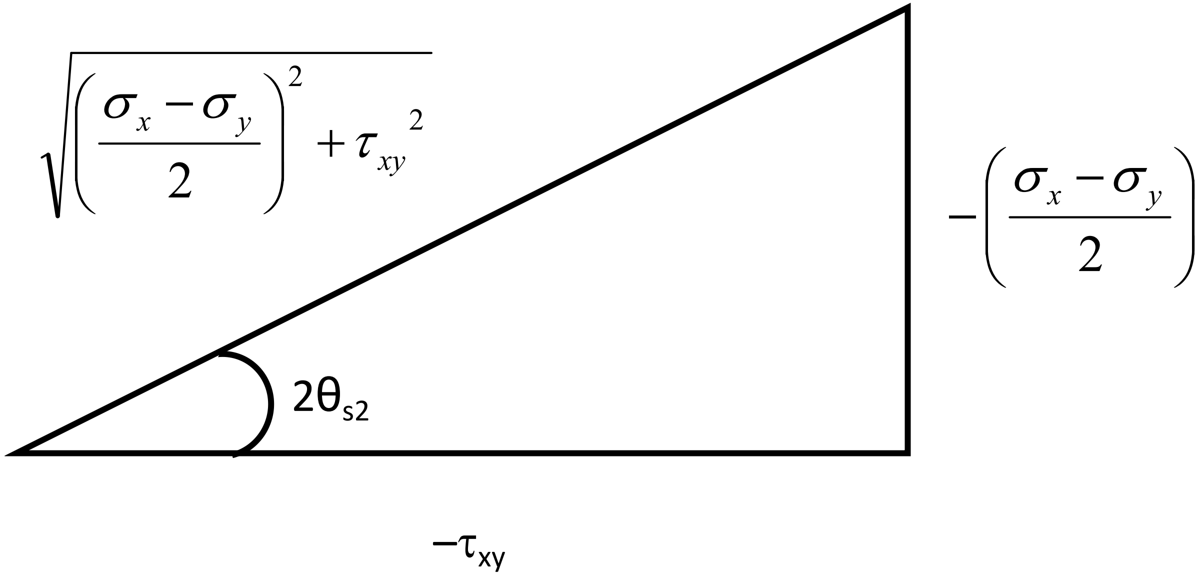
$$\tau_{xy}\frac{\sin 2\theta}{\cos 2\theta}.2=\sigma_x-\sigma_y$$

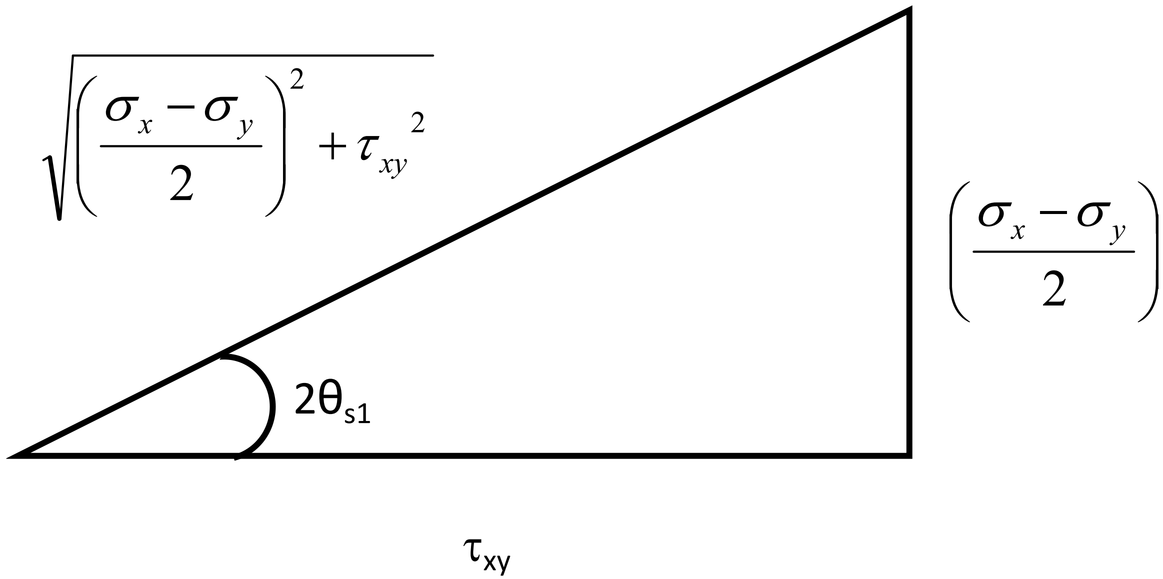
$$\tan 2\theta=\frac{1}{2}\frac{(\sigma_x-\sigma_y)}{\tau_{xy}} \qquad \rightarrow (14)$$

$$2\theta s=\tan^{-1}\left(\frac{\sigma_x-\sigma_y}{2\tau_{xy}}\right) \qquad \rightarrow (15)$$

Maximum shearing stress

This equation defines two values of $2\theta s$ differing by 180^o or two values of θs differing by 90^o. Corresponding to one of these two angles τ is algebraic maximum and to the second τ is algebraic minimum. The two values of the angles $2\theta s$ satisfying equation may be represented as shown.


$\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$
$-\left(\frac{\sigma_x - \sigma_y}{2}\right)$
$2\theta_{s2}$
$-\tau_{xy}$


$\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$
$\left(\frac{\sigma_x - \sigma_y}{2}\right)$
$2\theta_{s1}$
τ_{xy}

Where the upper sign refers to case 8a and lower to case 8b.

$$\sin 2\theta s_1 = \frac{\sigma_x - \sigma_y}{2R}$$

$$\sin 2\theta s_2 = -\frac{\sigma_x - \sigma_y}{2R}$$

$$\cos 2\theta s = \frac{\pm\tau_{xy}}{R}$$

Where the upper sign refers to case 1 and lower to case 2.

$$\cos 2\theta s_1 = \frac{\tau_{xy}}{R}$$

$$\cos 2\theta s_2 = -\frac{\tau_{xy}}{R}$$

Substituting these values of cosθs and sinθs in 2.

$$\tau_{\max} = \left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau_{\max} = \left(\frac{\sigma_x - \sigma_y}{2}\right)\frac{\sigma_x - \sigma_y}{2R} + \tau_{xy}.\frac{\tau_{xy}}{R}$$

$$\tau_{\max} = \frac{1}{R}\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2.\frac{1}{R}$$

$$\tau_{\max} = \frac{1}{R}\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2 = +\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$$

Positive sign represents the maximum shearing stress and the negative sign the minimum shearing stress.

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} \qquad \rightarrow (16)$$

$$\tau_{\min} = -\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2} \qquad \rightarrow (17)$$

Comparison of equation (7 & 14) shows that angles defined by these two equation have tangent that are negative reciprocal of each other and therefore angle (2θp) is 90^o away from (2θs). In other words Principal plane are inclined to the planes of maximum or minimum shear by a 45^o angle.

Normal stress on Maximum shear stress Element

In order to see if there is any normal stress existing on the element having maximum shear stress θs is to be substituted in 1.

$$\sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta s - \tau_{xy} \sin 2\theta s$$

$$\sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}(\sigma_x - \sigma_y) \times \frac{\tau_{xy}}{R} - \tau_{xy} . \frac{\sigma_x - \sigma_y}{2R}$$

$$\sigma = \frac{\sigma_x + \sigma_y}{2}, \qquad \rightarrow (18)$$

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \qquad \rightarrow (19)$$

Therefore it can be concluded that on the element on which τ_{max} occurs there will be a normal stress equal to the average of initial normal stresses.

Relationship between τ_{max} and Principal stress.

$$\sigma_{\max} = \frac{\sigma_x + \sigma_y}{2} + R \qquad \rightarrow \text{Major Principal stress}$$

$$\sigma_{\min} = \frac{\sigma_x + \sigma_y}{2} - R \qquad \rightarrow \text{Major Principal stress}$$

$$\tau_{\max} = R$$

$$\sigma_{\max} - \sigma_{\min} = 2R$$

$$\sigma_1 - \sigma_2 = 2R$$

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2}$$

Major or Minor Principal stresses.

when both tensile or compressive Stresses act together, the tensile stresses are always the major principal stresses irrespective of numerical values.

If both principal stresses are compressive the one with smaller magnitude is the major principal stress.

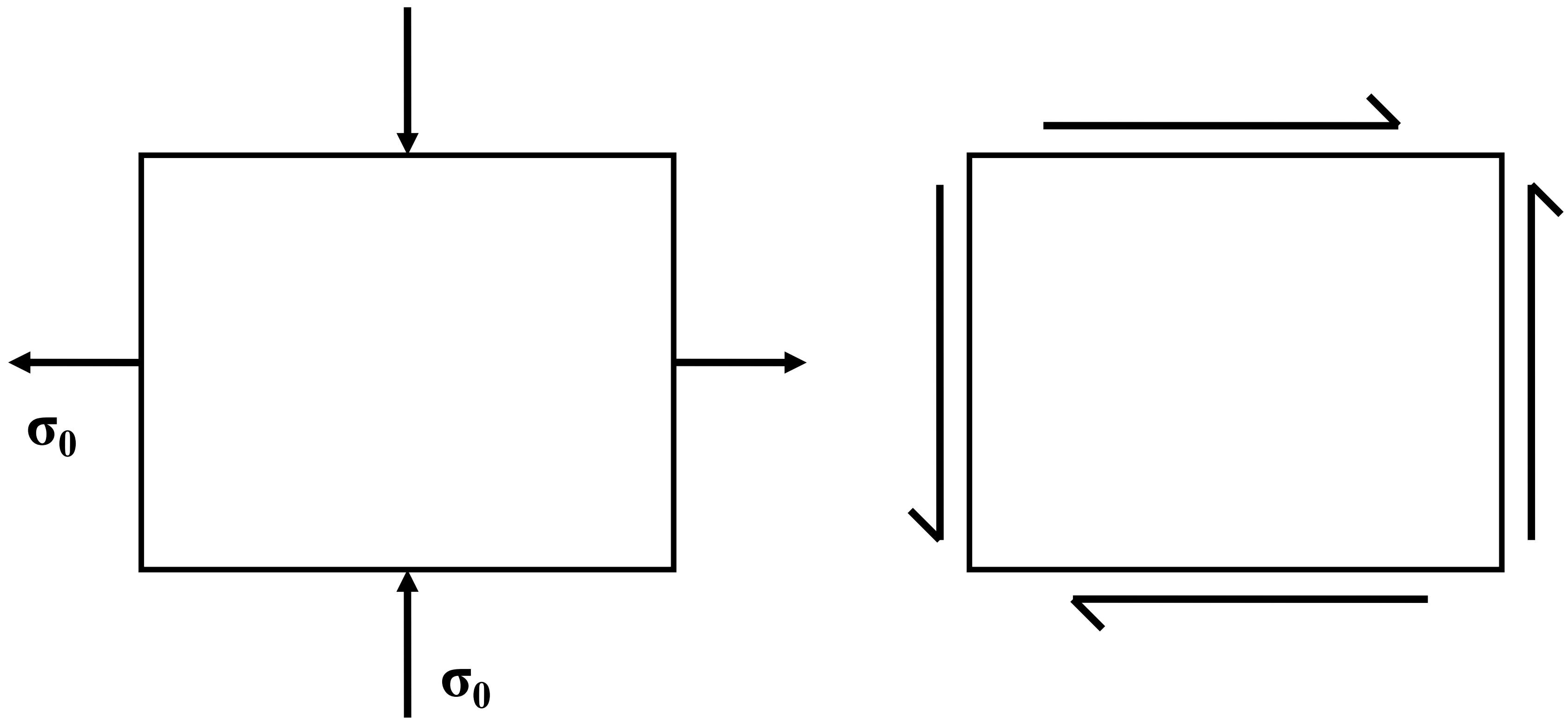
Pure shear.

If both σ_x or σ_y are equal in magnitude but of opposite sense then

$\sigma_x = -\sigma_y = \sigma_0$

$\tau_{max} = \sigma_0$

Then this element is subjected to pure shear at an angle of 45°.


σ_0
σ_0